###### Enineering Economics

###### TYPES OF INTEREST

* Interest rate: the ratio of interest chargeable at the end of a period of time, usually a year or less, and the money owed at the beginning of the period

i = I/P

Where I = interest earned

P = present amount or principal

i = interest rate per period

Example

If you borrow $1000 from me, and agree to pay the $1000 plus an extra $125 at the end of a year, what is the interest rate?

i = I/P, I = $125, P = $1000

i = 125/1000 = 0.125 = 12.5%

* Simple Interest
* Interest is paid only on the original amount, not on any accumulated interest

IN = PiN, N: number of interest periods (usually years)

FN = P + IN = P (1 + iN), F: Future worth

* rarely used (some bonds)

###### Example

## How much interest is due on a $1400 loan for 3 years at 10% simple interest per year?

I = PiN = $1400 x 0.10 x 3 = $420.00

* Compound Interest
* Interest itself accrues interest

F1 = P (1+i) where F1 = future worth at the end of the first year

F2 = F1 x (1 + i) = P (1 + i)2

F3 = F2 x (1 + i) = P (1 + i)3

FN = FN-1 x (1 + i) = P (1 + i)N-1 x (1 + i) = P (1 + i)N

###### Example

The power of interest: how long will it take for a penny to become a million dollars?

----at 12%? ----at 18%?

End of Year @ 12% @ 18%

1 $ 0.01 $0.01

10 $0.03 $0.05

25 $0.17 $0.63

80 $86.58 $5630.67

###### Compound Interest Factors

## Two types of factors

* Those that can be used to convert a single payment into a present or future value
* Those that can be used to convert a series of payments into a present or future value,

given that:

* the cash flows are at regular intervals
* the cash flows are uniform, or can be converted to uniform amounts.

Only the case for cash flows at discrete intervals, not continuous time, will be considered.

NOTE: The purpose of the factors is to simplify the calculations for certain forms of

cash flows. If a set of cash flows is not in a suitable form, we still calculate the

present value of each individual transaction (using a spreadsheet).

Nomenclature

The abbreviation for a factor is (X/Y, i, n)

Where: X is to be determined given Y, i, n

X and Y assume one of the values:

P – present value

F – future value

A – annuity (uniform flows)

Y can also be a G (gradient) if X is an A i is the effective interest rate per period (expressed in percent)

n is the number of periods

PARENTHETIC NOTE (on effective interest rate)

A nominal interest rate r of 8 percent quarterly, e.g., indicates an interest charge of 2 percent per quarter compound four times per year. If m is the number of compounding periods per year, the equivalent effective interest rate, or actual interest earned or paid, i from a nominal rate is



Interpretation of a factor: Given an initial amount P, find the future value if P is invested for 3 years at 5%.

Multiply the amount P by the factor (F/P, 5, 3); from p. A-49 (text), (F/P, 5, 3) = 1.1576 (single payment) (compare: F3 = P(1 +0.05)3 ⇒ F3/P = 1.1576 )

Derivation of the factors, with some examples

1. (F/P, *i*, n) single sum compound interest

F1 = P (1 + *i*) where F1 = future value the end of year 1

F2 = F1 x (1 + *i*) = P (1 + *i*)2

Fn = P (1 + *i*)n = P (F/P, *i*, n) **➀**

(F/P, i, n) = (1 + i)n

So the factor

Example: If $1000 is deposited in the bank at 7% for 10 years

P = $1000, *i* = 7%, n = 10 years

F = P (F/P, 7, 10) = 1000 (1.96715) = $1967.l5

The factor (F/P, i,n) can be calculated or looked up in an interest table. Spreadsheets also contain these factors as built-in functions.

2. (P/F, *i*, n) single sum present worth

Recall F = P(1 + *i*)n ⇒P = 

⇒ 

Note that (P/F, *i*, n) = 

Example:

How much work should be put in the bank now at 9%/year to accumulate $5000 in 8 years?

F = $5,000, *i* = 9%, n = 8 years

P = F (P/F, 9,8) = 5000 (0.50l9) = $2,509.35

3. (F/A, *i*, n) uniform series compound amount

Uniform transactions of amount A, each year for n years, starting at the end of year 1.

The future value of all these cashflows calculated to the end of year n, F is given by.

Fn = A [(1 + *i*)n-1] + A [(1 + *i*)n-2] + ……. +A [(1 + *i*)] + A

= A [(1 + *i*)n-1+ (1 + *i*)n-2 + ……. + [(1 + *i*) + 1] **➁**

That is, the last transaction at the end of year n earns no interest, the preceding transaction earns interest for 1 year, etc. and the first transaction compounds for n-1 years.

Multiply both sides of **➁** by (1+*i*)

⇒ Fn (1 + *i*) = A [(1 + *i*)n +(1 + *i*)n +(1 + *i*)n-1 + ……. + [(1 + *i*)2 + (1 + *i*)] **➂**

Subtracting **➁** from **➂**

⇒ Fn *i* = A [(1 + *i*)n – 1]

Fn=  **➃**

So: (

Example:

If $1,000 is deposited each year in the bank at 7% for 10 years, beginning in year 1:

A = $1000 *i* = 7% n = 10 years

F = A (, 7, 10) = 1000  = $13,816

*4*.  Sinking Fund

From **➃** ⇒ A = Fn /

A = Fn 

So 

Example:

How much should be deposited in the bank each year at 4% starting in year 1 to accumulate $10,000 at 4% starting in year 1 to accumulate $10,000 in 5 years?

F = $10,000 *i* = 4% n = 5 years A = ?

A = 10,000 ( = 10,000 (0.1846) = $1,846

*5*.  uniform series present worth

A constant amount A is deposited at the end of each year for n years. The present value for all these transactions is:

P =  **➄**

Multiply both sides of **➄** by (1+*i*)-1

*  **➅**

Subtract **➄** from **➅**

= A

P

P

P = A 

P = A 

So 

Example:

How much are 2 annual (end-of-period) payments of $500 each equivalent to today? (*i* = 10%)

A = $500 *i* = 10% n = 2 years

P = A $867.75

*6*.  Capital recovery factor

(A/P, i,n) = 

Example:

What 3 equal annual payments are required to fully reimburse a loan of $10,000? (*i* =10%)

P = $10,000 *i* = 10% n = 3 years

A = P 

*7*.  arithmetic gradient factor

Given a series of transactions at regular intervals, beginning in one period.

A1, A1+G, A1+2G, A1 +3G, …….

The sequence is based on a series of uniform cash flows (the A1) and a gradient G which can be converted to an equivalent uniform series, A". Calculating the future value of the growth part of the transactions alone:

F = G [(1 + *i*)n-2] + 2G [(1 + *i*)n-3] + ……. +(n-1)G **➆**

Multiplying both sides of **➆** by (1 + *i*)

* F (1 + *i*) = G [(1+ *i*)n-1] + 2G [(1+*i*)n-2] + ……. + (n-1) G (1 + *i*) **➇**

Subtract **➆** from **➇**

* F (1 + *i*) – F = G [(1 + *i*)n-1] + ( 1 + *i*)n-2 + ……. + (1 + *i*) + 1] – nG

## F*i* = G (F/A, *i*, n) – nG

Multiply both sides by (A/F, *i*, n):

* F*i* (A/F, *i*, n) = G [1 – n (A/F, *i*, n)]
* *i*A"= G [1-n(A/F, *i*, n)]

## But F (A/F, *i*, n) = A"

A" = G [1-n(A/F, *i*, n)] /*i*

A" = 

The above equivalent, A", is for the gradient part of the series.

For the entire cash flow series, an equivalent is given by

A = A' + A"

Example: A' = $400 G = $200 *i* = 10% n = 5 years

Find the present value.

A" = 200 (A/G, 10, 5) = 200 (1.8101) = $362.02

A = A' + A" = 400 +362 = $762

P = A (P/A, 10, 5) = 762 (3.7908) = $2,888

Note: The same procedure applies if the gradient is negative.

|  |  |  |  |
| --- | --- | --- | --- |
| **FACTOR** | **FORMULA** | **NAME** | **EXAMPLES**  **(ALL UNIFORM SERIES ASSUME END-OF-PERIOD PAYMENTS; *i* = 5%)** |
| (P/F, *i*, n) |  | Single Sum present worth | A company wants to have $1000 eight years from now. What amount is needed now to provide for it?  **Answer: 676.84** |
| (F/P, *i*, n) | (1 + *i*)n | Single Sum Compound amount | A firm borrows $1000 for 5 years. How much must it repay in a lump sum at the end of the 5th year?  **Answer: $1,276** |
| (F/A, *i*, n) |  | Uniform Series compound amount | If 4 annual deposits of $2000 each are placed in an account how much money would have accumulated immediately after the last deposit?  **Answer: $8,620** |
| (A/F, *i*, n) |  | Sinking Fund | How much should be deposited each in an account in order to accumulate $10,000 at the time of the fifth annual deposit?  **Answer: $1,809.70** |
| (P/A, *i*, n) |  | Uniform Series present worth | How much should be deposited in a fund to provide for 5 annual withdrawals of $100?  **Answer: $432.95** |
| (A/P, *i*, n) |  | Capital Recovery | What is the size of 10 equal annual payments to repay a loan of $1000? The first payment is 1 year after receiving the loan.  **Answer: $129.50** |
| (A/G, *i*, n) |  | Arithmetic Gradient | If $1000 is deposited after 1 year, and 9 more annual deposits are made increasing by $50 each time, what is the equivalent uniform annual amount?  **Answer: $304.95** |

Method to analyse a geometric series of transactions

If a series of transactions increases or decreases at a constant rate each period, the series is referred to as a geometric series.

This type of series applies, for example, when inflation affects our expenditures

Let g be the rate of change, each period (positive or negative)

And A1 be the first transaction (at the end of the 1st period )

P=  **➈**

Define a fictional growth rate: g1 = 

*  **➉**

Case:

1. If g > *i* or g < *i* , use **➉** where the factor P/A is simply based on g1. Note that if g > *i*, g1< 0 so the tables cannot be used to find the factor but it must be calculated (alternate method is shown below)
2. If g = *i*, it follows from **➈** that

P = 

Example:

Starting in a month, Peter will receive a monthly pension, where the first payment is for $2,000, indexed to the cost of living (increase of 0.3%/month). The interest rate is 4.5% compounded monthly. What is the present value of all of these revenues, which will continue for 5 years?

A1 = $2000, g = 0.003, *i* = ,

n = 5 x 12 = 60 months

So g1 = 

P = 2000 (P/A, 0.07477,60)/(1.003) = 2000 (58.6575)/(1.003)

P = $116,964

Degrees of Inflation

Mild - Annual price increases of 2 to 4 percent. Economy will prosper but business will want to

make larger profits during the period of growth and this then leads unions to bargain for

commensurate higher wages. The next degree of inflation then occurs.

Moderate - Price escalations of 5 to 9 percent. People then start purchasing more because they would

rather have more goods than money that is declining in value. Increased demand pulls

prices still higher and the next degree of inflation occurs.

Severe - Occurs when the annual rate reaches 10 percent or more. During double-digit inflation,

prices rise much faster than wages do . People on fixed incomes hurt badly. Only

debtors benefit – debts are repaid in less valuable dollars.

Hyperinflation - Rapid, uncontrolled inflation that destroys a nation’s economy. Here money becomes

essentially valueless, as the government prints it excessively to pay expenses, while

citizens go to a barter economy in which goods and services

are exchanged without currency.

Theories on the causes of inflation

1. Money mass increasing faster than actual production growth. If there is “too much” money available, the value falls ( supply and demand)
2. Production cost increases which are exaggerated in consumer prices.
3. Influences external to the economy (ex. OPEC oil prices affect most goods)
4. Inflexible prices because of unions (ex. Wage contracts restrain the market)
5. Inflation psychology: “buy right away” (i.e., when prices expected to rise)

###### Effects of inflation on economic analyses

A low inflation rate can be ignored for initial analyses since the effects on all options will be similar and the size of the effects will be reasonably small.

A high inflation should not be ignored because inflation will have dissimilar effects on different aspects of the options.

###### Two (equivalent) methods to include inflation in the calculate

1. Actual (or future) on current dollars, using the market interest (i): uses actual dollar flow.
2. Real (or Constant) dollars, using the real interest rate (i) uses cash flows expressed in purchasing power of a reference (or base) year (usually year 0)

In general: Real dollars = \* actual dollars AND

Actual dollars = real dollars \* (1+ F)n

Current Constant

Where n is the number of years from the reference (or base) year.

Now that we know how to express cash flows in actual dollars or real dollars we can use our formula to discount multiple cash flows to a single point in time to make comparisons. However, we must use the actual interest rate if our transactions are expressed in actual dollars, or use the real interest rate if the transactions are given in real dollars.

Real Interest Rate:

Let: i = actual interest rate (such as a bank rate )

(Market) Or MARR

F = inflation rate (average)

i = real interest rate (applied to real dollars)

Then: (1+i) = (1+F) (1+i1)or (1+i1) = (1+i)/(1+F)

That is, if your money grows at a value i, the REAL value of the growth in terms of increased purchasing power is LESS because of inflation. Therefore, the REAL rate of return expresses how much VALUE of your funds are growing.

Note: The following approximation, by expanding on the above formula:

i = (1+F)\*(1+i1)-1 = i1+F+i1F = i1+F (Since the order of magnitude of i1F is smaller)

Example: on: Use of the same basis for the cost of capital and cash flows

A proposal with an initial cost of $2,000 is expected to produce net returns of $850 per year for 3 years in real dollars. The minimum acceptable rate of return, based on the market cost of capital which includes inflation of 5 percent is 15 percent. Should the proposal be accepted?

If the cash flow were estimated in real dollars of constant purchasing power and the MARR included inflation adjustment, the proposal would be rejected because its present worth could be negative.

PW = - $2,000 + $850 (P/A , 15,3)

= -$2,000 + $850 (2.2832) = - $59

This evaluation unfairly penalizes the proposal because the MARR is based on the assumption that the cash flow will state the actual amount received each year. Real dollars can be converted to actual dollars by inflating them to an amount that is equivalent in purchasing power to their value today. For 5 percent annual inflation, the future cash flow equivalent to constant purchasing power of $850 a year is as follows:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **End of Year** | **Real dollars (no inflation)** | **X** | **5% Inflation** | **=** | **Actual dollars**  **(Inflated cash flow)** |
| 0 | -2,000 | X |  | = |  |
| 1 | 850 | X | 1.05 | = | 893 |
| 2 | 850 | X | (1.05)2 | = | 937 |
| 3 | 850 |  |  |  | 984 |

The inflated cash flow indicates that 3 years from now it would take $984 to acquire goods that could be purchased today for $850. When the inflated receipts are discounted at the inflation receipts are discounted at the inflation - adjusted MARR, the proposal has an acceptable present with:

PW = $2000 + $893 (P/F, 15,1) + $937 (P/F, 15,2) + $984 (P/F, 15,3)

= $2000 + $893 (0.8696) + $937 (0.7561) + $984 (0.6575)

= $132

This proposal merits approval