***Engineering Economics – Chapter 3***

Nominal and Effective Interest:

What is the effect of compounding interest more than once a year?

Let r = nominal interest rate per year

M = number of compounding periods per year

ieff = r/M = effective interest per period

Example: A $10,000 amount is deposited in the bank at a nominal rate of 10%/year, compounded quarterly (4 times a year).

Given: r = 10%/year (nominal); M = 4 periods per year; ieff = r/M = 10/4 = 2.5%/quarter = iQ

F3 = P(1 + iQ) = 10000(1 + 0.025) = 10250

F6 = F3(1 + iQ) = 10250(1.025) = 10506.25

Fq = F6(1 + iQ) = 10506.25(1.025) = 10768.91

F12 = Fq(1 + iQ) = 10768.91(1.025) = 11038.13

→ Interest accumulated at the end of 1 year is $1,038.13

This could also have been calculated as F12 = P(1 + iQ)4 = 10,000(1.025)4 = 10,000(F/P, 2.5, 4) = $11038.13

This is the same formula that has already been derived but there some key points.

1. The interest rate used in a formula must always be the effective interest rate per period.

2. The period length and compounding frequency must match.

i.e. if N is in months, the monthly effective rate must be used;

if N is in quarters, the quarterly effective rate must be used.

**THE NOMINAL INTEREST RATE IS ONLY FOR CALCULATING EFFECTIVE RATES.**

(note: also called APR – Annual Percentage Rate)

Effective Annual Rate

What is the effective annual rate ia on $10,000 compounded at a nominal rate of 10%/year, compounded quarterly? 10,000(1 + ia) = 11,038.13 → ia = 0.1038 = 10.38%

Note that (1 + ia) = (1 + iQ)4 → ia = (1 + iQ)4 – 1

In general: ia = (1 + r/M)M – 1.

• Interest represents the earning power of money

**TIME-VALUE EQUIVALENCE**

• Two things are equivalent when they produce the same effect.

- effective interest rate computed for a nominally stated interest rate is an equivalent expression of the interest charge since both interest charges produce the same effect on an investment.

• We differentiate between:

- Purchasing power of money: the amount of goods that can be purchased with a given sum of money. This varies up and down as a function of special localized circumstances and nationwide or worldwide economic conditions

- Earning power of money: this relates time and earnings to locate time-equivalent money amounts if $1000 were sealed and buried today, it would have a cash value of $1000 when it was dug up 2 years from now. Regardless of changes in buying power, the value remains constant because the earning power of the money was forfeited.

Consider $1000 deposited at 10% compounded annually. In two years, this amount will have a value of $1000(1 + 0.10)2 = $1210. Therefore, $1000 today is equivalent to $1210 in 2 years from now, if it earns interest at a prevailing rate of 10% compound yearly. Similarly, to have $1000 in 2 years from now, one needs only to deposit  today.

If 10% is an acceptable rate of return, an investor will be indifferent between having $826.45 in hand or having a trusted promise to receive $1000 in 2 years.

Consider further that the $1000 could have been used to pay two equal annual $500 installments. The buried $1000 could be retrieved after 1 year, an installment paid, and the remaining $500 reburied until the second repayment became due.

If the $1000 is deposited at 10% instead, $1100 would be available at the end of the first year. After the first $500 installment was paid, the remaining $600 would draw interest until the next payment. Paying the second $500 installment would leave $600(1.1) - $500 = $160 in the account.

Because of the earning power of money, the initial deposit could have been reduced to $868.77 to pay out $500 at the end of each of the 2 years:

First year: $868(1.10) - $500 = $955 - $500 = $455 ($454.55)

Second year: $455(1.10) = $500 = second installment.

The $868 is equivalent to $500 received 1 year from now plus another $500 received 2 years from now:

First year: $500/1.10 = $455

Second year: $455 + $500/(1.10)2 = $455 + $413 = $868

→ The annual effective interest can be calculated from the effective interest rate per period.

Continuous Compounding

In the limit where M tends to infinity, compounding is every instant. This approach may be used for enormous transactions (between banks or countries) and also for theoretical models in economics.



→ effective interest in the case of continuous compounding is calculated by:

iC = er – 1 (where r is the nominal rate)

Example: If r = 18.232% compounded continuously what is iC?

iC = e0.18232 – 1 = 0.20 or 20%

Example: If iC = 22.1%, what is the nominal rate r?

0.221 = er – 1

1.221 = er → ln (1.221) = r ln e

r = 0.20 or 20%

Example: Examine the effect of increased compounding frequency.

r(APR) = 10%; Discrete: ia = ; Continuous: iC = er - 1

|  |  |  |
| --- | --- | --- |
| **Frequency** | **M** | ia% |
| annual | 1 | 10.00 |
| semi-annual | 2 | 10.250 |
| quarterly | 4 | 10.381 |
| monthly | 12 | 10.471 |
| weekly | 52 | 10.506 |
| daily | 365 | 10.515 |
| continuous |  | 10.517 |

**PAYMENTS AND COMPOUNDING PERIODS MUST BE MADE COMPATIBLE BEFORE PERFORMING CALCULATIONS**

Case (1): Payments less frequent than compounding

Example: Three deposits of $2500 will be made every 2 years, starting in 2 years. If i = 10%/year how much will be in the bank in 6 years?

The F/A formula cannot be applied directly because the cash flow frequency is not synchronized with the compounding periods. Solving one term at a time:

F = 2500 [(F/P, 10, 4) + (F/P, 10, 2) + 1] = 2500 [(1.4641) + 1.21 + 1] = $9,185

Alternatively, an equivalent effective interest rate per 2-year period can be found

ieff = (1.10) 2 –1 = 0.21 = 21%

F = 2500(F/A, 21, 3) = 2500(3.6741) = $9,185

Example: You have a loan requiring payments of $500 every 6 months for the next 2 years. Assume that the next payment is due tomorrow. The interest rate is 12% compounded monthly. If you wish to pay off the loan now, how much do you owe?

[Concept: PV of remaining payments is what you owe now, i.e. both you and the lender are indifferent between PV and rest of payments]

PV = 500 + 500(P/A, is, 4) an effective semi-annual rate is is needed.

The effective monthly rate is: 

The effective semi-annual rate is: (1 + is) = (1 + im)6 → is = (1.01)6 – 1 = 0.06152 s-a

PV = 500 + 500 (P/A, 0.06152, 4) = $2,226.54

Case (2): Payments more frequent than compounding

Example: Payments of $50 are made at the end of each month, APR = 12% compounded quarterly what is F at the end of 1 year?

2 APPROACHES depending on assumptions:

(a) All payments accumulate to the end of compounding period (i.e., they are not earning interest until then) In this case $150 accumulates every quarter F = 150 (F/A, 3, 4) = $627.54

(b) Calculate an effective interest rate per period 1 + iQ = (1 + im)3 → im = (1.03)1/3 – 1 = 0.0099 F = 50 (F/A, 0.99, 12) = $633.79

The text assumes (b) unless otherwise stated

Varying Interest Rates

Must translate cash flows in stages, stopping at every boundary where the interest rate changes.

Example 1 P = $1000 F10 = ?

***8%***

***6%***

***4%***

0

10

9

8

7

6

5

4

3

2

1

F10 = 1000(1.04)4 (1.06)4 (1.08)2 = $1,722.68

What is the average 10-year yield?

(1 + i10)10 = (1.04)4 (1.06)4 (1.08)2 → i10 = 5.589%

Example 2 Find the equivalent annuity.

***8%***

***6%***

***5%***

0

6

5

4

3

2

1

600

700

1,000

600 (P/F, 5, 1) + 700 (P/F, 6, 1)(P/F, 5, 2) + 1000 (P/F, 7, 3)(P/F, 6, 1)(P/F, 5, 2)

= A[(P/A, 5, 2) + (P/F, 6, 1)(P/F, 5, 2) + (P/A, 7, 3)(P/F, 6, 1)(P/F, 5, 4)

→ A = $472.47

**CRITERIA FOR EVALUATING OPTIONS**

• Justification for choice of criteria

* Cash flows which occur at different points in time cannot be directly compared → an equivalent amount at some point in time, such as a present value to combine cash flows, has to be calculated.

- Several criteria are available to evaluate a proposal. The main ones are:

• Present Worth (PW) – (Chapter 4)

• Future Worth (FW) – (Chapter 4)

• Annual Worth (AW) – (Chapter 5)

• Internal Rate of Return (IRR) – (Chapter 6)

- A particular criterion might be the best one for a certain situation while in other situations, a combination of several criteria may be appropriate.

- The use of the Present Worth analysis is the easiest and most common approach but other criteria also provided useful information.

**SOME USEFUL CONCEPTS**

• Minimum Acceptable (Attractive) Rate of Return (MARR)

Every company or individual must select a suitable interest rate to evaluate the time value of money. Most companies establish a minimum acceptable (attractive) rate of return (MARR) for internal calculations. It is affected by many factors, such as:

- prevailing market (i.e., bank) rates

- cost of capital

- risk of business

- historical rates earned in the company

Example: A company ("JIVE-T" in Halifax is considering buying a new machine for bagging their chips. One possible model costs $120,000 with an estimated salvage value of $5,000 at the end of a ten year useful life. Given the anticipated growth in sales, the expected savings are $15,000 in the first year, increasing by $5,000 in subsequent years. Operating costs are $10,000/year. The MARR is 12%.

$5,000 - salvage

G = $5,000/year

$15,000

$60,000

10

0

-$10,000

-$120,000

• Project Screening

Certain simple techniques can be used to ascertain whether a project is viable, before detailed analysis is done. Since uncertainty increases tremendously the further into the future we go, one common requirement is that an investment generates enough revenues to cover the initial cost within a reasonable time frame. The criterion is called the payback period.

• Payback Period

(a) Without interest (faster, but inaccurate) Given Ft, the cash flow in period t, the payback period corresponds to the smallest value of n for which: 

(b) With interest (discounted payback period) the payback period requires that the PV (Present Value) of revenues surpass the PV of expenses. Therefore, the smallest value of n which satisfies:  is the payback period.

Example: "JIVE-T" payback:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  | ***Without interest*** | ***With interest*** |
| ***N*** | ***Outflows*** | ***Inflows*** | ***Net flow*** | ***Sum (t = 0 ton)*** | ***PV(t = 0 ton)*** |
| 0 | $(120,000) | 0 | $(120,000) | $(120,000) | $(120,000) |
| 1 | $(10,000) | $15,000 | $5,000 | $(115,000) | $(115,535) |
| 2 | $(10,000) | $20,000 | $10,000 | $(105,000) | $(107,564) |
| 3 | $(10,000) | $25,000 | $15,000 | $(90,000) | $(96,887) |
| 4 | $(10,000) | $30,000 | $20,000 | $(70,000) | $(84,177) |
| 5 | $(10,000) | $35,000 | $25,000 | $(45,000) | $(69,991) |
| 6 | $(10,000) | $40,000 | $30,000 | $(15,000) | $(54,722) |
| 7 | $(10,000) | $45,000 | $35,000 | $20,000 | $(38,960) |
| 8 | $(10,000) | $50,000 | $40,000 | $60,000 | $(22,805) |
| 9 | $(10,000) | $55,000 | $45,000 | $105,000 | $(6,577) |
| 10 | $(10,000) | $65,000 | $55,000 | $160,000 | $11,131 |

Therefore, using the undiscounted method suggests that the investment begins paying back after year 6, whereas if discounting is included, it isn't until year 10 that the project becomes viable.

Drawbacks to the Payback Period Method

An important limitation of this method is that it does not account for cash flows occurring after the payback period. For example, assume that in year 11 of the "JIVE-T" example, there is an overhaul cost of $100,000 and then revenues are constant. Overall, the project would not be viable. Because of this deficiency, the method should not be used as the sole decision-making criterion.

**PRESENT VALUE**

To see if the "JIVE-T" project is a good investment

PV = -$120,000 + [$15,000 + $15,000(A/G, 12, 10) - $10,000](P/A, 12, 10) + $5,000(P/F, 12, 10)

(3.5847) (5.6502) (0.3222)

PV = $11,132

Conclusion: Since the PV is greater than zero, the project is economical. Note that even if PV = 0 exactly, we accept the project, because the investment returns 12%. The fact that PV > 0 shows that they make a surplus above 12%.

**INVESTMENT POOL CONCEPT**: to explain the meaning of PW > 0

• assumption: funds not used in this project can earn MARR elsewhere in the company

Example: $120,000 would be worth FV(@12%) = $372,702 in 10 years if NOT invested in "JIVE-T". If instead the $120,000 WERE invested in "JIVE-T", then the FV(@MARR) of the net cash flows for the 10 years ($120,000 + $11,132)(1.12)10 = $407,276. Therefore investing in "JIVE-T" earns us an additional $34,574 by year 10. The present value of this amount is $11,132! Therefore, once can see that a positive PV indicates a gain over and above the MARR.

**CONDITIONS FOR PRESENT-WORTH COMPARISONS**

• Cash flows are known.

• Cash flows are in constant-value dollars.

• The interest rate is known.

• Comparisons are made with before-tax cash flows.

• Comparisons do not include intangible considerations.

• Comparisons do not include consideration of the availability of funds to implement alternatives.

**EQUIVALENT ANNUAL WORTH (AW)**

AW = $120,000(A/P, 12, 10) + $15,000 = $5,000(A/G, 12, 10) - $10,000 + $5,000(A/F, 12, 10)

(0.1770) (3.5847) (0.0570)

AW = $1968/year

Naturally, this same result is obtained directly by starting with the present value

AW = PV(A/P, 12, 10) = $11,132(0.1770) = $1970

It is evident that the AW is positive if the PV is positive and vice versa. If one is negative, the other is also negative. Therefore, the two criteria always give consistent results.

**FUTURE VALUE**

The same reasoning applies when the future value criterion is used. FV is also a consistent measure of project viability because it differs from the PV (or AW) by a positive constant.

**Note:** 

**INTERNAL RATE OF RETURN (IRR)**

Given all of the foreseen cash flows and their occurrence, we can calculate the only unknown for which the PV (or FW or AW) equals zero, i. The value of i, if it exists gives the effective interest earned (or charged) on money. That is, if I lend $100 for a month and demand a reimbursement of $150 at the end of the month, the internal rate of return is 50% per month.

PV = $100 = FV(P/V, i, 1) = $150(P/F, i, 1) =  ($150) → (1 + i)1 =  = 1.5 → i = 0.5 = 50%

The same reasoning applies for a more complicated set of transactions. For example examining the "JIVE‑T" problem, the IRR can be calculated by solving for:

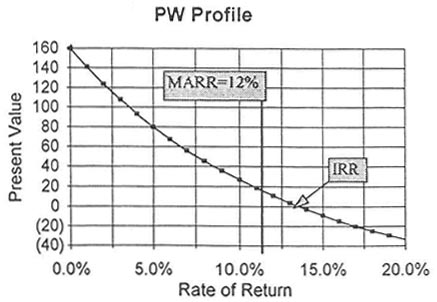
PV = 0 = -$120,000[$15,000 + $5,000(A/G, i, 10) - $10,000](P/A, i, 10) + $5,000(P/F, i, 10)

Solving the non-linear equation for the unknown i by trial and error, or using software such as Excel, we find i = 13.6%.

Conclusion: Since the MARR = 12%, this project is viable because IRR > MARR.

**CAUTION**: We would expect all of these methods to lead to the same conclusion. However, difficulties can arise because of the non-linear nature of the PV equation when solving for i (i.e, multiple roots). This difficulty will be considered shortly.

**PW PROFILE** (see also Figure 4.19, p. 24 Text)

To better understand the meaning of the IRR, consider the PW of the cash flow series as a function of i. Two points are easy to place to place on the graph: the IRR is the horizontal intercept (by definition); and the PV (@ i = 0) is the sum of the cash flows. Note what happens as i → ∞. The curve asymptotically approaches the initial cash flow at time zero! Why?

We can see that since the IRR > MARR, the project is acceptable. As long as this PW profile applies to the problem at hand, the IRR criteria and PW criteria lead to the same conclusion in a straight-forward manner. Exceptions to this will be reviewed later.

We also see that the PW > 0 at MARR, as expected.

Hints to save time when calculating IRR by trial and error

1. Identify the most important elements in a complicated calculations, and estimate an initial approximate interest rate based on these values.

2. Add the receipts and expenses without discounting (i.e., i = 0). If the sum is negative the IRR is less than zero (for an investment type of analysis). If the sum is positive, its value may suggest a reasonable guess for IRR to start.

3. If the salvage value is close to the initial cost, and there is an annuity, A/P = i.

4. The rule of 72: a sum doubles in value about every 72/i years.

5. Since interest rates are often less than 25%, an initial guess of 10% is reasonable.

6. Obviously, a financial calculator or spreadsheet is best.

Rule of 72:

This is a handy rule-of-thumb which provides a useful approximation to the time required to double money: n = 72/i.

Where does this rule come from? Starting with the formula for (F/P, i, n): F = P(1 + i)n

To double money (F = 2P): 2P = P(1 + i)n → Simplifying: 2 = (1 + i)n

ln(2) = n ln(1 + i) → n = ln(2)/ln(1 + i)

ln(2) = 0.6931 and ln(1 + i) ≈ i for small i (see graph) → n ≈ 0.72/i = 72/i %

Note: 72 gives a better approximation than 70.

Example 1:

(a) If we deposit $1,000 in the bank at 7% for 10 years: P = $1,000, i = 7%, n = 10, F = P(F/P, 7, 10) = 1,000(1.96715) = $1967.15.

(b) How much must we put in the bank now at 9%/year to accumulate $5,000 in 8 years? F = $5,000, i = 9%, n = 8, P = F(P/F, 9, 8) = $5,000(0.5019) = $2,509,39

Example 2:

N = 5 years, P = $80,000, F = $150,000

revenue: $1,500/year (= A+)

expenses: $850/year (= A-)

To find i: P = F(P/F, i, N) + (A+-A-)(P/A, i, N) → $80,000 = $150,000(P/F, i, 5) + $(1,500 – 850)(P/A, i, 5)

$650

Using the rule of 72:

Since $150,000 is almost the double of $80,000, if we ignore the annuity, we get i ≈ 14.4% ( = 72/5). So use initial estimate of i = 14%.

$150,000(P/F, 14, 5) + $650(P/A, 14, 5) = $80,136

0.51937 3.433

Since the results is greater than $80,000, we try a higher rate (to reduce the PV of the future transactions). So for i = 15%:

$150,000(P/F, 15, 5) + $650(P/A, 15, 5) = $76,756

0.4972 3.3522