Interest Rate Risk

##### Chapter Outline

##### Introduction

##### Duration

##### A General Formula for Duration

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* The Duration of a Zero-Coupon Bond
* The Duration of a Consol Bond (Perpetuities)

##### Features of Duration

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* Duration and Yield
* Duration and Coupon Interest

##### The Economic Meaning of Duration

* Semiannual Coupon Bonds

##### Duration and Immunization

* Duration and Immunizing Future Payments
* Immunizing the Whole Balance Sheet of an FI

##### Immunization and Regulatory Considerations

**Summary**

**Appendix 9A: Difficulties in Applying the Duration Model to Real-World FI Balance Sheets**

* Duration Matching can be Costly
* Immunization is a Dynamic Problem
* Large Interest Rate Changes and Convexity
* The Problem of the Flat Term Structure
* Floating-Rate Loans and Bonds
* Demand Deposits and Passbook Savings
* Mortgages and Mortgage-Backed Securities
* Futures, Options, Swaps, Caps, and Other Contingent Claims

Solutions for End-of-Chapter Questions and Problems: Chapter Nine

1. What are the two different general interpretations of the concept of **duration**, and what is the technical definition of this term? How does duration differ from maturity?

Duration measures the average life of an asset or liability in economic terms. As such, duration has economic meaning as the interest sensitivity (or interest elasticity) of an asset’s value to changes in the interest rate. Duration differs from maturity as a measure of interest rate sensitivity because duration takes into account the time of arrival and the rate of reinvestment of all cash flows during the assets life. Technically, duration is the weighted-average time to maturity using the relative present values of the cash flows as the weights.

2. Two bonds are available for purchase in the financial markets. The first bond is a 2-year, $1,000 bond that pays an annual coupon of 10 percent. The second bond is a 2-year, $1,000, zero-coupon bond.

a. What is the duration of the coupon bond if the current yield-to-maturity (YTM) is 8 percent? 10 percent? 12 percent? (Hint: You may wish to create a spreadsheet program to assist in the calculations.)

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Coupon Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Annual payments | |  |
|  | YTM = | 0.08 |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $100.00 | 0.92593 | $92.59 |  | $92.59 |  |  |  |  |  |
| 2 | $1,100.00 | 0.85734 | $943.07 |  | $1,886.15 |  |  |  |  |  |
|  |  | Price = | $1,035.67 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,978.74 | Duration = |  | 1.9106 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | YTM = | 0.10 |  |  |  |  |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $100.00 | 0.90909 | $90.91 |  | $90.91 |  |  |  |  |  |
| 2 | $1,100.00 | 0.82645 | $909.09 |  | $1,818.18 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,909.09 | Duration = |  | 1.9091 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | YTM = | 0.12 |  |  |  |  |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $100.00 | 0.89286 | $89.29 |  | $89.29 |  |  |  |  |  |
| 2 | $1,100.00 | 0.79719 | $876.91 |  | $1,753.83 |  |  |  |  |  |
|  |  | Price = | $966.20 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,843.11 | Duration = |  | 1.9076 | = Numerator/Price | |

b. How does the change in the current YTM affect the duration of this coupon bond?

Increasing the yield-to-maturity decrease the duration of the bond.

c. Calculate the duration of the zero-coupon bond with a YTM of 8 percent, 10 percent, and 12 percent.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Zero Coupon Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.00 |  |  |  |  |
|  | YTM = | 0.08 |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $0.00 | 0.92593 | $0.00 |  | $0.00 |  |  |  |  |  |
| 2 | $1,000.00 | 0.85734 | $857.34 |  | $1,714.68 |  |  |  |  |  |
|  |  | Price = | $857.34 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,714.68 | Duration = |  | 2.0000 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | YTM = | 0.10 |  |  |  |  |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $0.00 | 0.90909 | $0.00 |  | $0.00 |  |  |  |  |  |
| 2 | $1,000.00 | 0.82645 | $826.45 |  | $1,652.89 |  |  |  |  |  |
|  |  | Price = | $826.45 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,652.89 | Duration = |  | 2.0000 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | YTM = | 0.12 |  |  |  |  |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $0.00 | 0.89286 | $0.00 |  | $0.00 |  |  |  |  |  |
| 2 | $1,000.00 | 0.79719 | $797.19 |  | $1,594.39 |  |  |  |  |  |
|  |  | Price = | $797.19 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,594.39 | Duration = |  | 2.0000 | = Numerator/Price | |

d. How does the change in the current YTM affect the duration of the zero-coupon bond?

Changing the yield-to-maturity does not affect the duration of the zero coupon bond.

e. Why does the change in the YTM affect the coupon bond differently than the zero-coupon bond?

Increasing the YTM on the coupon bond allows for a higher reinvestment income that more quickly recovers the initial investment. The zero-coupon bond has no cash flow until maturity.

3. A one-year, $100,000 loan carries a market interest rate of 12 percent. The loan requires payment of accrued interest and one-half of the principal at the end of 6 months. The remaining principal and accrued interest are due at the end of the year.

a. What is the duration of this loan?

Cash flow in 6 months = $100,000 x .12 x .5 + $50,000 = $56,000 interest and principal.

Cash flow in 1 year = $50,000 x 1.06 = $53,000 interest and principal.

Time Cash Flow PVIF CF\*PVIF T\*CF\*CVIF

1 $56,000 0.943396 $52,830.19 $52,830.19

2 $53,000 0.889996 $47,169.81 $94,339.62

Price = $100,000.00 $147,169.81 = Numerator

years

b. What will be the cash flows at the end of 6 months and at the end of the year?

Cash flow in 6 months = $100,000 x .12 x .5 + $50,000 = $56,000 interest and principal.

Cash flow in 1 year = $50,000 x 1.06 = $53,000 interest and principal.

c. What is the present value of each cash flow discounted at the market rate? What is the total present value?

$56,000 ÷ 1.06 = $52,830.19 = PVCF1

$53,000 ÷ (1.06)2 = $47,169.81 = PVCF2

=$100,000.00 = PV Total CF

d. What proportion of the total present value of cash flows occurs at the end of 6 months? What proportion occurs at the end of the year?

Proportiont=.5 = $52,830.19 ÷ $100,000 x 100 = 52.830 percent.

Proportiont=1 = $47,169.81 ÷ $100,000 x 100 = 47.169 percent.

e. What is the weighted-average life of the cash flows on the loan?

D = 0.5283 x 0.5 years + 0.47169 x 1.0 years = 0.26415 + 0.47169 = 0.73584 years.

f. How does this weighted-average life compare to the duration calculated in part (a) above?

The two values are the same.

4. What is the duration of a five-year, $1,000 Treasury bond with a 10 percent semiannual coupon selling at par? Selling with a YTM of 12 percent? 14 percent? What can you conclude about the relationship between duration and yield to maturity? Plot the relationship. Why does this relationship exist?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Five-year Treasury Bond | | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Semiannual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $50.00 | 0.95238 | $47.62 |  | $23.81 |  |  | PVIF = 1/(1+YTM/2)^(Time\*2) | | |
| 1 | $50.00 | 0.90703 | $45.35 |  | $45.35 |  |  |  |  |  |
| 1.5 | $50.00 | 0.86384 | $43.19 |  | $64.79 |  |  |  |  |  |
| 2 | $50.00 | 0.8227 | $41.14 |  | $82.27 |  |  |  |  |  |
| 2.5 | $50.00 | 0.78353 | $39.18 |  | $97.94 |  |  |  |  |  |
| 3 | $50.00 | 0.74622 | $37.31 |  | $111.93 |  |  |  |  |  |
| 3.5 | $50.00 | 0.71068 | $35.53 |  | $124.37 |  |  |  |  |  |
| 4 | $50.00 | 0.67684 | $33.84 |  | $135.37 |  |  |  |  |  |
| 4.5 | $50.00 | 0.64461 | $32.23 |  | $145.04 |  |  |  |  |  |
| 5 | $1,050.00 | 0.61391 | $644.61 |  | $3,223.04 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $4,053.91 | Duration = |  | 4.0539 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | Five-year Treasury Bond | | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Semiannual payments | | |
|  | YTM = | 0.12 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $50.00 | 0.9434 | $47.17 |  | $23.58 |  |  | Duration | YTM |  |
| 1 | $50.00 | 0.89 | $44.50 |  | $44.50 |  |  | 4.0539 | 0.10 |  |
| 1.5 | $50.00 | 0.83962 | $41.98 |  | $62.97 |  |  | 4.0113 | 0.12 |  |
| 2 | $50.00 | 0.79209 | $39.60 |  | $79.21 |  |  | 3.9676 | 0.14 |  |
| 2.5 | $50.00 | 0.74726 | $37.36 |  | $93.41 |  |  |  |  |  |
| 3 | $50.00 | 0.70496 | $35.25 |  | $105.74 |  |  |  |  |  |
| 3.5 | $50.00 | 0.66506 | $33.25 |  | $116.38 |  |  |  |  |  |
| 4 | $50.00 | 0.62741 | $31.37 |  | $125.48 |  |  |  |  |  |
| 4.5 | $50.00 | 0.5919 | $29.59 |  | $133.18 |  |  |  |  |  |
| 5 | $1,050.00 | 0.55839 | $586.31 |  | $2,931.57 | . |  |  |  |  |
|  |  | Price = | $926.40 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $3,716.03 | Duration = |  | 4.0113 | = Numerator/Price | |

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | | Five-year Treasury Bond | | |  |  |  |  |  |  |  |
|  | | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Semiannual payments | | |
|  | | YTM = | 0.14 |  |  | Maturity = | 5 |  |  |  |  |
| Time | | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | | $50.00 | 0.93458 | $46.73 |  | $23.36 |  |  |  |  |  |
| 1 | | $50.00 | 0.87344 | $43.67 |  | $43.67 |  |  |  |  |  |
| 1.5 | | $50.00 | 0.8163 | $40.81 |  | $61.22 |  |  |  |  |  |
| 2 | | $50.00 | 0.7629 | $38.14 |  | $76.29 |  |  |  |  |  |
| 2.5 | | $50.00 | 0.71299 | $35.65 |  | $89.12 |  |  |  |  |  |
| 3 | | $50.00 | 0.66634 | $33.32 |  | $99.95 |  |  |  |  |  |
| 3.5 | | $50.00 | 0.62275 | $31.14 |  | $108.98 |  |  |  |  |  |
| 4 | | $50.00 | 0.58201 | $29.10 |  | $116.40 |  |  |  |  |  |
| 4.5 | | $50.00 | 0.54393 | $27.20 |  | $122.39 |  |  |  |  |  |
| 5 | | $1,050.00 | 0.50835 | $533.77 |  | $2,668.83 |  |  |  |  |  |
|  | |  | Price = | $859.53 |  |  |  |  |  |  |  |
|  |  |  | Numerator =  As the yield to maturity increases, duration decreases because of the reinvestment of interim cash flows at higher rates. | | $3,410.22 | Duration = |  | 3.9676 | = Numerator/Price | |

5. Consider three Treasury bonds which each have 10 percent semiannual coupons and trade at par.

a. Calculate the duration for a bond that has a maturity of 4 years, 3 years, and 2 years?

Please see the calculations on the next page.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| a. | Four-year Treasury Bond | | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Semiannual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 4 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $50.00 | 0.952381 | $47.62 |  | $23.81 | PVIF = 1/(1+YTM/2)^(Time\*2) | | | | |
| 1 | $50.00 | 0.907029 | $45.35 |  | $45.35 |  |  |  |  |  |
| 1.5 | $50.00 | 0.863838 | $43.19 |  | $64.79 |  |  |  |  |  |
| 2 | $50.00 | 0.822702 | $41.14 |  | $82.27 |  |  |  |  |  |
| 2.5 | $50.00 | 0.783526 | $39.18 |  | $97.94 |  |  |  |  |  |
| 3 | $50.00 | 0.746215 | $37.31 |  | $111.93 |  |  |  |  |  |
| 3.5 | $50.00 | 0.710681 | $35.53 |  | $124.37 |  |  |  |  |  |
| 4 | $1,050.00 | 0.676839 | $710.68 |  | $2,842.73 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $3,393.19 | Duration = | | 3.3932 | = Numerator/Price | |
|  |  |  |  | |  |  | |  |  | |
|  | Three-year Treasury Bond | | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Semiannual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 3 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $50.00 | 0.952381 | $47.62 |  | $23.81 | PVIF = 1/(1+YTM/2)^(Time\*2) | | | | |
| 1 | $50.00 | 0.907029 | $45.35 |  | $45.35 |  |  |  |  |  |
| 1.5 | $50.00 | 0.863838 | $43.19 |  | $64.79 |  |  |  |  |  |
| 2 | $50.00 | 0.822702 | $41.14 |  | $82.27 |  |  |  |  |  |
| 2.5 | $50.00 | 0.783526 | $39.18 |  | $97.94 |  |  |  |  |  |
| 3 | $1,050.00 | 0.746215 | $783.53 |  | $2,350.58 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $2,664.74 | Duration | = | 2.6647 | = Numerator/Price | |
|  |  |  |  | |  |  |  |  |  | |
|  | Two-year Treasury Bond | | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Semiannual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $50.00 | 0.952381 | $47.62 |  | $23.81 | PVIF = 1/(1+YTM/2)^(Time\*2) | | | | |
| 1 | $50.00 | 0.907029 | $45.35 |  | $45.35 |  |  |  |  |  |
| 1.5 | $50.00 | 0.863838 | $43.19 |  | $64.79 |  |  |  |  |  |
| 2 | $1,050.00 | 0.822702 | $863.84 |  | $1,727.68 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,861.62 | Duration | = | 1.8616 | = Numerator/Price | |

b. What conclusions can you reach about the relationship of duration and the time to maturity? Plot the relationship.

As maturity decreases, duration decreases at a decreasing rate. Although the graph below does not illustrate with great precision, the change in duration is less than the change in time to maturity.

|  |  |  |
| --- | --- | --- |
|  |  | Change in |
| Duration | Maturity | Duration |
| 1.8616 | 2 |  |
| 2.6647 | 3 | 0.8031 |
| 3.3932 | 4 | 0.7285 |

6. A six-year, $10,000 CD pays 6 percent interest annually. What is the duration of the CD? What would be the duration if interest were paid semiannually? What is the relationship of duration to the relative frequency of interest payments?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Six-year CD |  |  |  |  |  |  |  |  |  |
|  | Par value = | $10,000 |  |  | Coupon = | 0.06 |  | Annual payments | | |
|  | YTM = | 0.06 |  |  | Maturity = | 6 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $600.00 | 0.94340 | $566.04 |  | $566.04 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $600.00 | 0.89000 | $534.00 |  | $1,068.00 |  |  |  |  |  |
| 3 | $600.00 | 0.83962 | $503.77 |  | $1,511.31 |  |  |  |  |  |
| 4 | $600.00 | 0.79209 | $475.26 |  | $1,901.02 |  |  |  |  |  |
| 5 | $600.00 | 0.74726 | $448.35 |  | $2,241.77 |  |  |  |  |  |
| 6 | $10,600 | 0.70496 | $7,472.58 |  | $44,835.49 |  |  |  |  |  |
|  |  | Price = | $10,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $52,123.64 | Duration | = | 5.2124 | = Numerator/Price | |

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Six-year CD |  |  |  |  |  |  |  |  |  |
|  | Par value = | $10,000 |  |  | Coupon = | 0.06 |  | Semiannual payments | | |
|  | YTM = | 0.06 |  |  | Maturity = | 6 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $300.00 | 0.970874 | $291.26 |  | $145.63 | PVIF = 1/(1+YTM/2)^(Time\*2) | | | | |
| 1 | $300.00 | 0.942596 | $282.78 |  | $282.78 |  |  |  |  |  |
| 1.5 | $300.00 | 0.915142 | $274.54 |  | $411.81 |  |  |  |  |  |
| 2 | $300.00 | 0.888487 | $266.55 |  | $533.09 |  |  |  |  |  |
| 2.5 | $300.00 | 0.862609 | $258.78 |  | $646.96 |  |  |  |  |  |
| 3 | $300.00 | 0.837484 | $251.25 |  | $753.74 |  |  |  |  |  |
| 3.5 | $300.00 | 0.813092 | $243.93 |  | $853.75 |  |  |  |  |  |
| 4 | $300.00 | 0.789409 | $236.82 |  | $947.29 |  |  |  |  |  |
| 4.5 | $300.00 | 0.766417 | $229.93 |  | $1,034.66 |  |  |  |  |  |
| 5 | $300.00 | 0.744094 | $223.23 |  | $1,116.14 |  |  |  |  |  |
| 5.5 | $300.00 | 0.722421 | $216.73 |  | $1,192.00 |  |  |  |  |  |
| 6 | $10,300 | 0.701380 | $7,224.21 |  | $43,345.28 |  |  |  |  |  |
|  |  | Price = | $10,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $51,263.12 | Duration | = | 5.1263 | = Numerator/Price | |

Duration decreases as the frequency of payments increases. This relationship occurs because (a) cash is being received more quickly, and (b) reinvestment income will occur more quickly from the earlier cash flows.

7. What is the duration of a consol bond that sells at a YTM of 8 percent? 10 percent? 12 percent? What is a consol bond? Would a consol trading at a YTM of 10 percent have a greater duration than a 20-year zero-coupon bond trading at the same YTM? Why?

A consol is a bond that pays a fixed coupon each year forever. A consol Consol Bond

trading at a YTM of 10 percent has a duration of 11 years, while a zero- YTM D = 1 + 1/R

coupon bond trading at a YTM of 10 percent, or any other YTM, has a 0.08 13.50 years

duration of 20 years because no cash flows occur before the twentieth 0.10 11.00 years

year. 0.12 9.33 years

8. Maximum Pension Fund is attempting to balance one of the bond portfolios under its management. The fund has identified three bonds which have five-year maturities and which trade at a YTM of 9 percent. The bonds differ only in that the coupons are 7 percent, 9 percent, and 11 percent.

a. What is the duration for each bond?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Five-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.07 |  | Annual payments | | |
|  | YTM = | 0.09 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $70.00 | 0.917431 | $64.22 |  | $64.22 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $70.00 | 0.841680 | $58.92 |  | $117.84 |  |  |  |  |  |
| 3 | $70.00 | 0.772183 | $54.05 |  | $162.16 |  |  |  |  |  |
| 4 | $70.00 | 0.708425 | $49.59 |  | $198.36 |  |  |  |  |  |
| 5 | $1,070.00 | 0.649931 | $695.43 |  | $3,477.13 |  |  |  |  |  |
|  |  | Price = | $922.21 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $4,019.71 | Duration | = | 4.3588 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | Five-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.09 |  | Annual payments | | |
|  | YTM = | 0.09 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $90.00 | 0.917431 | $82.57 |  | $82.57 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $90.00 | 0.841680 | $75.75 |  | $151.50 |  |  |  |  |  |
| 3 | $90.00 | 0.772183 | $69.50 |  | $208.49 |  |  |  |  |  |
| 4 | $90.00 | 0.708425 | $63.76 |  | $255.03 |  |  |  |  |  |
| 5 | $1,090.00 | 0.649931 | $708.43 |  | $3,542.13 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $4,239.72 | Duration | = | 4.2397 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | Five-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.11 |  | Annual payments | | |
|  | YTM = | 0.09 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $110.00 | 0.917431 | $100.92 |  | $100.92 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $110.00 | 0.841680 | $92.58 |  | $185.17 |  |  |  |  |  |
| 3 | $110.00 | 0.772183 | $84.94 |  | $254.82 |  |  |  |  |  |
| 4 | $110.00 | 0.708425 | $77.93 |  | $311.71 |  |  |  |  |  |
| 5 | $1,110.00 | 0.649931 | $721.42 |  | $3,607.12 |  |  |  |  |  |
|  |  | Price = | $1,077.79 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $4,459.73 | Duration | = | 4.1378 | = Numerator/Price | |

b. What is the relationship between duration and the amount of coupon interest that is paid? Plot the relationship.

Duration decreases as the amount of coupon interest increases.

Change in

Duration Coupon Duration

* 1. 7%

4.2397 9% -0.1191

4.1378 11% -0.1019

9. An insurance company is analyzing three bonds and is using duration as the measure of interest rate risk. The three bonds all trade at a YTM of 10 percent and have $10,000 par values. The bonds differ only in the amount of annual coupon interest that they pay: 8, 10, or 12 percent.

a. What is the duration for each five-year bond?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Five-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $10,000 |  |  | Coupon = | 0.08 |  | Annual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $800.00 | 0.909091 | $727.27 |  | $727.27 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $800.00 | 0.826446 | $661.16 |  | $1,322.31 |  |  |  |  |  |
| 3 | $800.00 | 0.751315 | $601.05 |  | $1,803.16 |  |  |  |  |  |
| 4 | $800.00 | 0.683013 | $546.41 |  | $2,185.64 |  |  |  |  |  |
| 5 | $10,800.00 | 0.620921 | $6,705.95 |  | $33,529.75 |  |  |  |  |  |
|  |  | Price = | $9,241.84 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $39,568.14 | Duration | = | 4.2814 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | Five-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $10,000 |  |  | Coupon = | 0.10 |  | Annual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $1,000.00 | 0.909091 | $909.09 |  | $909.09 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $1,000.00 | 0.826446 | $826.45 |  | $1,652.89 |  |  |  |  |  |
| 3 | $1,000.00 | 0.751315 | $751.31 |  | $2,253.94 |  |  |  |  |  |
| 4 | $1,000.00 | 0.683013 | $683.01 |  | $2,732.05 |  |  |  |  |  |
| 5 | $11,000.00 | 0.620921 | $6,830.13 |  | $34,150.67 |  |  |  |  |  |
|  |  | Price = | $10,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $41,698.65 | Duration | = | 4.1699 | = Numerator/Price | |

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Five-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $10,000 |  |  | Coupon = | 0.12 |  | Annual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $1,200.00 | 0.909091 | $1,090.91 |  | $1,090.91 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $1,200.00 | 0.826446 | $991.74 |  | $1,983.47 |  |  |  |  |  |
| 3 | $1,200.00 | 0.751315 | $901.58 |  | $2,704.73 |  |  |  |  |  |
| 4 | $1,200.00 | 0.683013 | $819.62 |  | $3,278.46 |  |  |  |  |  |
| 5 | $11,200.00 | 0.620921 | $6,954.32 |  | $34,771.59 |  |  |  |  |  |
|  |  | Price = | $10,758.16 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $43,829.17 | Duration | = | 4.0740 | = Numerator/Price | |

b. What is the relationship between duration and the amount of coupon interest that is paid?

Duration decreases as the amount of coupon interest increases.

Change in

Duration Coupon Duration 4.2814 7%

4.1699 9% -0.1115

4.0740 11% -0.0959

10. You can obtain a loan for $100,000 at a rate of 10 percent for two years. You have a choice of either paying the principal at the end of the second year or amortizing the loan, that is, paying interest and principal in equal payments each year. The loan is priced at par.

a. What is the duration of the loan under both methods of payment?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Two-year loan: Principal and interest at end of year two. | | | | | |  |  |  |  |
|  | Par value = | 100,000 |  |  | Coupon = | 0.00 |  | No annual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $0.00 | 0.90909 | $0.00 |  | $0.00 |  |  | PVIF = 1/(1+YTM)^(Time) | | |
| 2 | $121,000 | 0.82645 | $100,000.0 |  | 200,000.00 |  |  |  |  |  |
|  |  | Price = | $100,000.0 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | 200,000.00 | Duration | = | 2.0000 | = Numerator/Price | |
|  | Two-year loan: Interest at end of year one, P & I at end of year two. | | | | | | | |  |  |
|  | Par value = | 100,000 |  |  | Coupon = | 0.10 |  | Annual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $10,000 | 0.909091 | $9,090.91 |  | $9,090.91 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $110,000 | 0.826446 | $90,909.09 |  | 181,818.18 |  |  |  |  |  |
|  |  | Price = | $100,000.0 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | 190,909.09 | Duration | = | 1.9091 | = Numerator/Price | |
|  |  |  |  |  |  |  |  |  |  |  |
|  | Two-year loan: Amortized over two years. | | | | | Amortized payment of $57.619.05 | | | | |
|  | Par value = | 100,000 |  |  | Coupon = | 0.10 |  |  |  |  |
|  | YTM = | 0.10 |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $57,619.05 | 0.909091 | $52,380.95 |  | $52,380.95 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $57,619.05 | 0.826446 | $47,619.05 |  | $95,238.10 |  |  |  |  |  |
|  |  | Price = | $100,000.0 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | 147,619.05 | Duration | = | 1.4762 | = Numerator/Price | |

b. Explain the difference in the two results?

Duration decreases dramatically when a portion of the principal is repaid at the end of year one. Duration often is described as the weighted-average maturity of an asset. If more weight is given to early payments, the effective maturity of the asset is reduced.

Repayment Change in

Duration Provisions Duration

2.0000 [P&I@2](mailto:P&I@2)

1.9091 [I@1,P&I@2](mailto:I@1,P&I@2) -0.0909

1.4762 Amortize -0.4329

11. How is duration related to the interest elasticity of a fixed-income security? What is the relationship between duration and the price of the fixed-income security?

Taking the first derivative of a bond’s (or any fixed-income security) price (P) with respect to the yield to maturity (R) provides the following:



The economic interpretation is that D is a measure of the percentage change in price of a bond for a given percentage change in yield to maturity (interest elasticity). This equation can be rewritten to provide a practical application:



In other words, if duration is known, then the change in the price of a bond due to small changes

in interest rates, R, can be estimated using the above formula.

12. You have discovered that the price of a bond rose from $975 to $995 when the YTM fell from 9.75 percent to 9.25 percent. What is the duration of the bond?

We know 

13. Calculate the duration of a 2-year, $1,000 bond that pays an annual coupon of 10 percent and trades at a yield of 14 percent. What is the expected change in the price of the bond if interest rates decline by 0.50 percent (50 basis points)?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Two-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Annual payments | | |
|  | YTM = | 0.14 |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $100.00 | 0.87719 | $87.72 |  | $87.72 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $1,100.00 | 0.76947 | $846.41 |  | $1,692.83 |  |  |  |  |  |
|  |  | Price = | $934.13 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,780.55 | Duration | = | 1.9061 | = Numerator/Price | |

Expected change in price = . This implies a new price of $941.94. The actual price using conventional bond price discounting would be $941.99. The difference of $0.05 is due to convexity, which was not considered in this solution.

14. The duration of an 11-year, $1,000 Treasury bond paying a 10 percent semiannual coupon and selling at par has been estimated at 6.9 years.

a. What is the modified duration of the bond (Modified Duration = D/(1 + R))?

MD = 6.9/(1 + .10/2) = 6.57 years

b. What will be the estimated price change of the bond if market interest rates increase 0.10 percent (10 basis points)? If rates decrease 0.20 percent (20 basis points)?

Estimated change in price = -MD x ΔR x P = -6.57 x 0.001 x $1,000 = -$6.57.

Estimated change in price = -MD x ΔR x P = -6.57 x -0.002 x $1,000 = $13.14.

c. What would be the actual price of the bond under each rate change situation in part (b) using the traditional present value bond pricing techniques? What is the amount of error in each case?

Rate Price Actual

Change Estimated Price Error

+ 0.001 $993.43 $993.45 $0.02

- 0.002 $1,013.14 $1,013.28 -$0.14

15. Suppose you purchase a five-year, 13.76 percent bond that is priced to yield 10 percent.

a. Show that the duration of this annual payment bond is equal to four years.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Five-year Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.1376 |  | Annual payments | | |
|  | YTM = | 0.10 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $137.60 | 0.909091 | $125.09 |  | $125.09 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $137.60 | 0.826446 | $113.72 |  | $227.44 |  |  |  |  |  |
| 3 | $137.60 | 0.751315 | $103.38 |  | $310.14 |  |  |  |  |  |
| 4 | $137.60 | 0.683013 | $93.98 |  | $375.93 |  |  |  |  |  |
| 5 | $1,137.60 | 0.620921 | $706.36 |  | $3,531.80 |  |  |  |  |  |
|  |  | Price = | $1,142.53 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $4,570.40 | Duration | = | 4.0002 | = Numerator/Price | |

b. Show that, if interest rates rise to 11 percent within the next year and that if your investment horizon is four years from today, you will still earn a 10 percent yield on your investment.

Value of bond at end of year four: PV = ($137.60 + $1,000) ÷ 1.11 = $1,024.86.

Future value of interest payments at end of year four: $137.60\*FVIFn=4, i=11% = $648.06.

Future value of all cash flows at n = 4:

Coupon interest payments over four years $550.40

Interest on interest at 11 percent 97.66

Value of bond at end of year four $1,024.86

Total future value of investment $1,672.92

Yield on purchase of asset at $1,142.53 = $1,672.92\*PVIVn=4, i=?%  ⇒ i = 10.002332%.

c. Show that a 10 percent yield also will be earned if interest rates fall next year to 9 percent.

Value of bond at end of year four: PV = ($137.60 + $1,000) ÷ 1.09 = $1,043.67.

Future value of interest payments at end of year four: $137.60\*FVIFn=4, i=9% = $629.26.

Future value of all cash flows at n = 4:

Coupon interest payments over four years $550.40

Interest on interest at 9 percent 78.86

Value of bond at end of year four $1,043.67

Total future value of investment $1,672.93

Yield on purchase of asset at $1,142.53 = $1,672.93\*PVIVn=4, i=?%  ⇒ i = 10.0025 percent.

16. Consider the case where an investor holds a bond for a period of time longer than the duration of the bond, that is, longer than the original investment horizon.

a. If market interest rates rise, will the return that is earned exceed or fall short of the original required rate of return? Explain.

In this case the actual return earned would exceed the yield expected at the time of purchase. The benefits from a higher reinvestment rate would exceed the price reduction effect if the investor holds the bond for a sufficient length of time.

b. What will happen to the realized return if market interest rates decrease? Explain.

If market rates decrease, the realized yield on the bond will be less than the expected yield because the decrease in reinvestment earnings will be greater than the gain in bond value.

c. Recalculate parts (b) and (c) of problem 15 above, assuming that the bond is held for all five years, to verify your answers to parts (a) and (b) of this problem.

The case where interest rates rise to 11 percent, n = five years:

Future value of interest payments at end of year five: $137.60\*FVIFn=5, i=11% = $856.95.

Future value of all cash flows at n = 5:

Coupon interest payments over five years $688.00

Interest on interest at 11 percent 168.95

Value of bond at end of year five $1,000.00

Total future value of investment $1,856.95

Yield on purchase of asset at $1,142.53 = $1,856.95\*PVIFn=5, i=?%  ⇒ i = 10.2012 percent.

The case where interest rates fall to 9 percent, n = five years:

Future value of interest payments at end of year five: $137.60\*FVIFn=5, i=9% = $823.50.

Future value of all cash flows at n = 5:

Coupon interest payments over five years $688.00

Interest on interest at 9 percent 135.50

Value of bond at end of year five $1,000.00

Total future value of investment $1,823.50

Yield on purchase of asset at $1,142.53 = $1,823.50\*PVIVn=5, i=?%  ⇒ i = 9.8013 percent.

d. If either calculation in part (c) is greater than the original required rate of return, why would an investor ever try to match the duration of an asset with his investment horizon?

The answer has to do with the ability to forecast interest rates. Forecasting interest rates is a very difficult task, one that most financial institution money managers are unwilling to do. For most managers, betting that rates would rise to 11 percent to provide a realized yield of 10.20 percent over five years is not a sufficient return to offset the possibility that rates could fall to 9 percent and thus give a yield of only 9.8 percent over five years.

17. Two banks are being examined by the regulators to determine the interest rate sensitivity of their balance sheets. Bank A has assets composed solely of a 10-year, 12 percent, $1 million loan. The loan is financed with a 10-year, 10 percent, $1 million CD. Bank B has assets composed solely of a 7-year, 12 percent zero-coupon bond with a current (market) value of $894,006.20 and a maturity (principal) value of $1,976,362.88. The bond is financed with a 10-year, 8.275 percent coupon, $1,000,000 face value CD with a YTM of 10 percent. The loan and the CDs pay interest annually, with principal due at maturity.

a. If market interest rates increase 1 percent (100 basis points), how do the market values of the assets and liabilities of each bank change? That is, what will be the net affect on the market value of the equity for each bank?

For Bank A, an increase of 100 basis points in interest rate will cause the market values of assets and liabilities to decrease as follows:

Loan: $120\*PVIVAn=10,i=13% + $1,000\*PVIVn=10,i=13% = $945,737.57.

CD: $100\*PVIVAn=10,i=11% + $1,000\*PVIVn=10,i=11% = $941,107.68.

Therefore, the decrease in value of the asset was $4,629.89 less than the liability.

For Bank B:

Bond: $1,976,362.88\*PVIVn=7,i=13% = $840,074.08.

CD: $82.75\*PVIVAn=10,i=11% + $1,000\*PVIVn=10,i=11% = $839,518.43.

The bond value decreased $53,932.12, and the CD value fell $54,487.79. Therefore, the decrease in value of the asset was $555.67 less than the liability.

b. What accounts for the differences in the changes of the market value of equity between the two banks?

The assets and liabilities of Bank A change in value by different amounts because the durations of the assets and liabilities are not the same, even though the face values and maturities are the same. For Bank B, the maturities of the assets and liabilities are different, but the current market values and durations are the same. Thus the change in interest rates causes the same (approximate) change in value for both liabilities and assets.

c. Verify your results above by calculating the duration for the assets and liabilities of each bank, and estimate the changes in value for the expected change in interest rates. Summarize your results.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Ten-year CD:Bank B | | (Calculation in millions) | | |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.08 |  | Annual payments | | |
|  | **YTM =** | **0.10** |  |  | Maturity = | 10 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $82.75 | 0.909091 | $75.23 |  | $75.23 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $82.75 | 0.826446 | $68.39 |  | $136.78 |  |  |  |  |  |
| 3 | $82.75 | 0.751315 | $62.17 |  | $186.51 |  |  |  |  |  |
| 4 | $82.75 | 0.683013 | $56.52 |  | $226.08 |  |  |  |  |  |
| 5 | $82.75 | 0.620921 | $51.38 |  | $256.91 |  |  |  |  |  |
| 6 | $82.75 | 0.564474 | $46.71 |  | $280.26 |  |  |  |  |  |
| 7 | $82.75 | 0.513158 | $42.46 |  | $297.25 |  |  |  |  |  |
| 8 | $82.75 | 0.466507 | $38.60 |  | $308.83 |  |  |  |  |  |
| 9 | $82.75 | 0.424098 | $35.09 |  | $315.85 |  |  |  |  |  |
| 10 | $1,082.75 | 0.385543 | $417.45 |  | $4,174.47 |  |  |  |  |  |
|  |  | Price = | $894.006 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $6,258.15 | Duration | = | 7.0001 | = Numerator/Price | |

The duration for the CD of Bank B is calculated above to be 7.001 years. Since the bond is a zero-coupon, the duration is equal to the maturity of 7 years.

Using the duration formula to estimate the change in value:

Bond: ΔValue = 

CD: ΔValue = 

The difference in the change in value of the assets and liabilities for Bank B is $1,024.04 using the duration estimation model. The small difference in this estimate and the estimate found in part a above is due to the convexity of the two financial assets.

The duration estimates for the loan and CD for Bank A are presented below:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Ten-year Loan: Bank A | | | (Calculation in millions) | | | |
|  | Par value = | $1,000 | |  |  | Coupon = | | 0.12 |  | Annual payments | | |
|  | YTM = | 0.12 | |  |  | Maturity = | | 10 |  |  |  |  |
| Time | Cash Flow | PVIF | | PV of CF |  | PV\*CF\*T | |  |  |  |  |  |
| 1 | $120.00 | 0.892857 | | $107.14 |  | $107.14 | | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $120.00 | 0.797194 | | $95.66 |  | $191.33 | |  |  |  |  |  |
| 3 | $120.00 | 0.711780 | | $85.41 |  | $256.24 | |  |  |  |  |  |
| 4 | $120.00 | 0.635518 | | $76.26 |  | $305.05 | |  |  |  |  |  |
| 5 | $120.00 | 0.567427 | | $68.09 |  | $340.46 | |  |  |  |  |  |
| 6 | $120.00 | 0.506631 | | $60.80 |  | $364.77 | |  |  |  |  |  |
| 7 | $120.00 | 0.452349 | | $54.28 |  | $379.97 | |  |  |  |  |  |
| 8 | $120.00 | 0.403883 | | $48.47 |  | $387.73 | |  |  |  |  |  |
| 9 | $120.00 | 0.360610 | | $43.27 |  | $389.46 | |  |  |  |  |  |
| 10 | $1,120.00 | 0.321973 | | $360.61 |  | $3,606.10 | |  |  |  |  |  |
|  |  | Price = | | $1,000.00 |  |  | |  |  |  |  |  |
|  |  |  | | Numerator = | | $6,328.25 | | Duration | = | 6.3282 | = Numerator/Price | |

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Ten-year CD: Bank A | | (Calculation in millions) | | |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.10 |  | Annual payments | | |
|  | **YTM =** | **0.10** |  |  | Maturity = | 10 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $100.00 | 0.909091 | $90.91 |  | $90.91 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $100.00 | 0.826446 | $82.64 |  | $165.29 |  |  |  |  |  |
| 3 | $100.00 | 0.751315 | $75.13 |  | $225.39 |  |  |  |  |  |
| 4 | $100.00 | 0.683013 | $68.30 |  | $273.21 |  |  |  |  |  |
| 5 | $100.00 | 0.620921 | $62.09 |  | $310.46 |  |  |  |  |  |
| 6 | $100.00 | 0.564474 | $56.45 |  | $338.68 |  |  |  |  |  |
| 7 | $100.00 | 0.513158 | $51.32 |  | $359.21 |  |  |  |  |  |
| 8 | $100.00 | 0.466507 | $46.65 |  | $373.21 |  |  |  |  |  |
| 9 | $100.00 | 0.424098 | $42.41 |  | $381.69 |  |  |  |  |  |
| 10 | $1,100.00 | 0.385543 | $424.10 |  | $4,240.98 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $6,759.02 | Duration | = | 6.7590 | = Numerator/Price | |

Using the duration formula to estimate the change in value:

Loan: ΔValue = 

CD: ΔValue = 

The difference in the change in value of the assets and liabilities for Bank A is $4,943.66 using the duration estimation model. The small difference in this estimate and the estimate found in part a above is due to the convexity of the two financial assets. The reason the change in asset values for Bank A is considerably larger than for Bank B is because of the difference in the durations of the loan and CD for Bank A.

18. If you use duration only to immunize your portfolio, what three factors affect changes in the net worth of a financial institution when interest rates change?

The change in net worth for a given change in interest rates is given by the following equation:



Thus, three factors are important in determining ΔE.

1) [*DA* - *D* *L* *k*] or the leveraged adjusted duration gap. The larger this gap, the more exposed is the FI to changes in interest rates.

2) *A,* or the size of the FI. The larger is *A*, the larger is the exposure to interest rate changes.

3) *R*/1 + *R*, or interest rate shocks. The larger is the shock, the larger is the exposure.

19. Financial Institution XY has assets of $1 million invested in a 30-year, 10 percent semiannual coupon Treasury bond selling at par. The duration of this bond has been estimated at 9.94 years. The assets are financed with equity and a $900,000, 2-year, 7.25 percent semiannual coupon capital note selling at par.

a. What is the leverage-adjusted duration gap of Financial Institution XY?

The duration of the capital note is 1.8975 years.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Two-year Capital Note | |  |  |  |  |  |  |  |  |
|  | Par value = | $900 |  |  | Coupon = | 0.0725 |  | Semiannual payments | | |
|  | **YTM =** | **0.0725** |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $32.63 | 0.965018 | $31.48 |  | $15.74 | PVIF = 1/(1+YTM/2)^(Time\*2) | | | | |
| 1 | $32.63 | 0.931260 | $30.38 |  | $30.38 |  |  |  |  |  |
| 1.5 | $32.63 | 0.898683 | $29.32 |  | $43.98 |  |  |  |  |  |
| 2 | $932.63 | 0.867245 | $808.81 |  | $1,617.63 |  |  |  |  |  |
|  |  | Price = | $900.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,707.73 | Duration | = | 1.8975 | = Numerator/Price | |

The leverage-adjusted duration gap can be found as follows:



b. What is the impact on equity value if the relative change in all market interest rates is a decrease of 20 basis points? Note, the relative change in interest rates is ΔR/(1+R/2) = -0.0020.

The change in net worth using leverage adjusted duration gap is given by:



c. Using the information that you calculated in parts (a) and (b), infer a general statement about the desired duration gap for a financial institution if interest rates are expected to increase or decrease.

If the FI wishes to be immune from the effects of interest rate risk, that is, either positive or negative changes in interest rates, a desirable leverage-adjusted duration gap (LADG) is zero. If the FI is confident that interest rates will fall, a positive LADG will provide the greatest benefit. If the FI is confident that rates will increase, then negative LADG would be beneficial.

d. Verify your inference by calculating the change in market value of equity assuming that the relative change in all market interest rates is an increase of 30 basis points.



e. What would the duration of the assets need to be to immunize the equity from changes in market interest rates?

Immunizing the equity from changes in interest rates requires that the LADG be 0. Thus, (DA-DLk) = 0 ⇒ DA = DLk, or DA = 0.9\*1.8975 = 1.70775 years.

20. The balance sheet for Gotbucks Bank, Inc. (GBI) is presented below ($ millions):

AssetsLiabilities and Equity

Cash $30 Core deposits $20

Federal funds 20 Federal funds 50

Loans (floating) 105 Euro CDs 130

Loans (fixed) 65 Equity 20

Total assets $220 Total liabilities & equity $220

NOTES TO THE BALANCE SHEET: The Fed funds rate is 8.5 percent, the floating loan rate is LIBOR + 4 percent, and currently LIBOR is 11 percent. Fixed rate loans have five-year maturities, are priced at par, and pay 12 percent annual interest. Core deposits are fixed-rate for 2 years at 8 percent paid annually. Euros currently yield 9 percent.

a. What is the duration of the fixed-rate loan portfolio of Gotbucks Bank?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Five-year Loan | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.1200 |  | Annual payments | | |
|  | YTM = | 0.12 |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $120.00 | 0.892857 | $107.14 |  | $107.14 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $120.00 | 0.797194 | $95.66 |  | $191.33 |  |  |  |  |  |
| 3 | $120.00 | 0.711780 | $85.41 |  | $256.24 |  |  |  |  |  |
| 4 | $120.00 | 0.635518 | $76.26 |  | $305.05 |  |  |  |  |  |
| 5 | $1,120.00 | 0.567427 | $635.52 |  | $3,177.59 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $4,037.35 | Duration | = | 4.0373 | = Numerator/Price | |

The duration is 4.037 years.

b. If the duration of the floating-rate loans and fed funds is 0.36 years, what is the duration of GBI’s assets?

DA = [30(0) + 65(4.037) + 125(.36)]/220 = 1.397 years

c. What is the duration of the core deposits if they are priced at par?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Two-year Core Deposits | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.08 |  | Annual payments | | |
|  | **YTM =** | **0.08** |  |  | Maturity = | 2 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $80.00 | 0.92593 | $74.07 |  | $74.07 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $1,080.00 | 0.85734 | $925.93 |  | $1,851.85 |  |  |  |  |  |
|  |  | Price = | $1,000.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,925.93 | Duration | = | 1.9259 | = Numerator/Price | |

The duration of the core deposits is 1.926 years.

d. If the duration of the Euro CDs and Fed funds liabilities is 0.401 years, what is the duration of GBI’s liabilities?

DL = [20\*(1.926) + 180\*(.401)]/200 = .5535 years

e. What is GBI’s duration gap? What is its interest rate risk exposure?

GBI’s leveraged adjusted duration gap is: 1.397 - 200/220 \* (.5535) = .8938 years

f. What is the impact on the market value of equity if the relative change in all market interest rates is an increase of 1 percent (100 basis points)? Note, the relative change in interest rates is (ΔR/(1+R)) = 0.01.

Since GBI’s duration gap is positive, an increase in interest rates will lead to a decline in net worth. For a 1 percent increase, the change in net worth is:

E = -0.8938 \* (0.01) \* $220 = -$1,966,360 (new net worth will be $18,033,640).

g. What is the impact on the market value of equity if the relative change in all market interest rates is a decrease of 0.5 percent (-50 basis points)?

Since GBI’s duration gap is positive, an decrease in interest rates will lead to an increase in net worth. For a 0.5 percent decrease, the change in net worth is:

E = -0.8938 \* (-0.005) \* $220 = $983,180 (new net worth will be $20,983,180).

f. What variables are available to GBI to immunize the bank? How much would each variable need to change to get DGAP equal to 0?

Immunization requires the bank to have a leverage-adjusted duration gap of 0.0. Therefore, GBI could reduce the duration of its assets to 0.5535 years by using more fed funds and floating rate loans. Or GBI could use a combination of reducing asset duration and increasing liability duration in such a manner that LADG is 0.0.

21. Hands Insurance Company issued a $90 million, 1-year, zero-coupon note at 8 percent add-on annual interest (paying one coupon at the end of the year). The proceeds were used to fund a $100 million, 2-year commercial loan at 10 percent annual interest. Immediately after these transactions were simultaneously closed, all market interest rates increased 1.5 percent (150 basis points).

a. What is the true market value of the loan investment and the liability after the change in interest rates?

The market value of the loan declined by $2,551,830.92 million, to $97.448 million.

MVA = $10,000,000\*PVIFAn=2, i=11.5% + $100,000,000\* PVIFn=2, i=11.5% = $97,448,169.08

The market value of the note declined $1,232,876.71 to $88.767 million.

MVL = $97,200,000\* PVIFn=1, i=9.5% = $88,767,123.29

b. What impact did these changes in market value have on the market value of the equity?

ΔE = ΔA - ΔL = -$2,551,830.92 – (-$1,232,876.71) = -$1,313,954.21.

The increase in interest rates caused the asset to decrease in value more than the liability which caused the value of the net worth to decrease by $1,313,954.21.

c. What was the duration of the loan investment and the liability at the time of issuance?

The duration of the loan investment is 1.909 years. Note: The calculation for this loan is shown in problem 2, second example. The duration of the liability is one year since it is a zero-coupon note.

d. Use these duration values to calculate the expected change in the value of the loan and the liability for the predicted increase of 1.5 percent in interest rates.

The approximate change in the market value of the loan for a 150 basis points change is:

. The expected market value of the loan using the above formula is $97,396,818.18, or $97.400 million.

The approximate change in the market value of the note for a 150 basis points change is:

. The expected market value of the note using the above formula is $88,750,000, or $88.750 million.

e. What was the duration gap of Hands Insurance Company after the issuance of the asset and note?

The leverage-adjusted duration gap was [1.909 – (0.9)1.0] = 1.009 years.

f. What was the change in equity value forecast by this duration gap for the predicted increase in interest rates of 1.5 percent?

ΔMVE = -1.009\*[0.015/(1.10)]\*$100,000,000 = -$1,375,909. Note that this calculation assumes that the change in interest rates is relative to the rate on the loan. Further, this estimated change in net worth compares with the estimates above in part (d) as follows:

ΔMVE = ΔMVA - ΔMVL = -$2,603,182 – (-$1,250,000) = -$1,353,182.

g. If the interest rate prediction had been available during the time period in which the loan and the liability were being negotiated, what suggestions would you offer to reduce the possible effect on the equity of the company? What are the difficulties in implementing your ideas?

Obviously the duration of the loan could be shortened relative to the liability, or the liability duration could be lengthened relative to the loan, or some combination of both. Shortening the loan duration would mean the possible use of variable rates, or some earlier payment of principal as was demonstrated in problem 10. The duration of the liability can not be lengthened without extending the maturity life of the note. In either case, the loan officer may have been up against market or competitive constraints in that the borrower or investor may have had other options. Other methods to reduce the interest rate risk under conditions of this nature include using derivatives such as options, futures, and swaps.

22. The following balance sheet information is available (amounts in $ thousands and duration in years) for a financial institution:

Amount Duration

T-bills $90 0.50

T-notes 55 0.90

T-bonds 176 *x*

Loans 2,274 7.00

Deposits 2,092 1.00

Federal funds 238 0.01

Equity 715

Treasury bonds are 5-year maturities paying 6 percent semiannually and selling at par.

a. What is the duration of the T-bond portfolio? 4.393 years as shown below.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Treasury Bond | |  |  |  |  |  |  |  |  |
|  | Par value = | $176 |  |  | Coupon = | 0.06 |  | Semiannual payments | | |
|  | **YTM =** | **0.06** |  |  | Maturity = | 5 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 0.5 | $5.28 | 0.97087 | $5.13 |  | $2.56 |  |  |  |  |  |
| 1 | $5.28 | 0.94260 | $4.98 |  | $4.98 |  |  |  |  |  |
| 1.5 | $5.28 | 0.91514 | $4.83 |  | $7.25 |  |  |  |  |  |
| 2 | $5.28 | 0.88849 | $4.69 |  | $9.38 |  |  |  |  |  |
| 2.5 | $5.28 | 0.86261 | $4.55 |  | $11.39 |  |  |  |  |  |
| 3 | $5.28 | 0.83748 | $4.42 |  | $13.27 |  |  |  |  |  |
| 3.5 | $5.28 | 0.81309 | $4.29 |  | $15.03 |  |  |  |  |  |
| 4 | $5.28 | 0.78941 | $4.17 |  | $16.67 |  |  |  |  |  |
| 4.5 | $5.28 | 0.76642 | $4.05 |  | $18.21 |  |  |  |  |  |
| 5 | $181.28 | 0.74409 | $134.89 |  | $674.45 | . |  |  |  |  |
|  |  | Price = | $176.00 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $773.18 | Duration | = | 4.3931 | = Numerator/Price | |

b. What is the average duration of all the assets?

[(.5)(90) + (.9)(55) + (4.393)(176) + (7)(2724)]/3045 = 6.55 years

c. What is the average duration of all the liabilities?

[(1)(2092) + (0.01)(238)]/2330 = 0.90 years

d. What is the leverage-adjusted duration gap? What is the interest rate risk exposure?

DG = DA - kDL = 6.55 - (2330/3045)(0.90) = 5.86 years

The duration gap is positive, indicating that an increase in interest rates will lead to a decline in net worth.

e. What is the forecast impact on the market value of equity caused by a relative upward shift in the entire yield curve of 0.5 percent [i.e., ΔR/(1+R) = 0.0050]?

The market value of the equity will change by the following:

MVE = -DG \* (A) \* R/(1 + R) = -5.86(3045)(0.0050) = -$89.22. The loss in equity of $89,220 will reduce the equity (net worth) to $625,780.

f. If the yield curve shifted downward by 0.25 percent (i.e., ΔR/(1+R) = -0.0025), what is the forecasted impact on the market value of equity?

The change in the value of equity is MVE = -5.86(3045)(-0.0025) = $44,610. Thus, the market value of equity (net worth) will increase by $44,610, to $759,610.

g. What variables are available to the financial institution to immunize the balance sheet? How much would each variable need to change to get DGAP equal to 0?

Immunization requires the bank to have a leverage-adjusted duration gap of 0.0. Therefore, the FI could reduce the duration of its assets to 0.6887 years by using more T-bills and floating rate loans. Or the FI could try to increase the duration of its deposits possibly by using fixed-rate CDs with a maturity of 3 or 4 years. Finally, the FI could use a combination of reducing asset duration and increasing liability duration in such a manner that LADG is 0.0. This duration gap of 5.86 years is quite large and it is not likely that the FI will be able to reduce it to zero by using only balance sheet adjustments. For example, even if the FI moved all of its loans into T-bills, the duration of the assets still would exceed the duration of the liabilities after adjusting for leverage. This adjustment in asset mix would imply foregoing a large yield advantage from the loan portfolio relative to the T-bill yields in most economic environments.

23. Assume that a goal of the regulatory agencies of financial institutions is to immunize the ratio of equity to total assets, that is, Δ(E/A) = 0. Explain how this goal changes the desired duration gap for the institution. Why does this differ from the duration gap necessary to immunize the total equity? How would your answers change to part (h) in problem 20, or part (g) in problem 22, if immunizing equity to total assets was the goal?

In this case the duration of the assets and liabilities should be equal. Thus if ΔE = ΔA, then by definition the leveraged adjusted duration gap is positive, since ΔE would exceed kΔA by the amount of (1 – k), and the FI would face the risk of increases in interest rates. In reference to problems 20 and 22, the adjustments on the asset side of the balance sheet would not need to be as strong, although the difference likely would not be large if the FI in question is a depository institution such as a bank or S&L.

The following questions and problems are based on material in the appendix to the chapter.

24. Identify and discuss three criticisms of using the duration model to immunize the portfolio of a financial institution.

The three criticisms are:

a Immunization is a dynamic problem because duration changes over time. Thus, it is necessary to rebalance the portfolio as the duration of the assets and liabilities change over time.

b Duration matching can be costly because it is not easy to restructure the balance sheet periodically, especially for large FIs.

c Duration is not an appropriate tool for immunizing portfolios when the expected interest rate changes are large because of the existence of convexity. Convexity exists because the relationship between bond price changes and interest rate changes is not linear, which is assumed in the estimation of duration. Using convexity to immunize a portfolio will reduce the problem.

25. In general, what changes have occurred in the financial markets that allow financial institutions to more rapidly and efficiently restructure their balance sheets to meet desired goals? Why is it critical for an investment manager who has a portfolio immunized to match a desired investment horizon to rebalance the portfolio periodically? Why is convexity a desirable feature to be captured in a portfolio of assets? What is convexity?

The growth of purchased funds markets, asset securitization, and loan sales markets have increased considerably the speed of major balance sheet restructurings. Further, as these markets have developed, the cost of the necessary transactions has also decreased. Finally, the growth and development of the derivative markets provides significant alternatives to managing the risk of interest rate movements only with on-balance sheet adjustments.

Assets approach maturity at a different rate of speed than the duration of the same assets approaches zero. Thus, after a period of time, a portfolio or asset that was immunized against interest rate risk will no longer be immunized. In fact, portfolio duration will exceed the remaining time in the investment or target horizon, and changes in interest rates could prove costly to the institution.

Convexity is a property of fixed-rate assets that reflects nonlinearity in the reflection of price-rate relationships. This characteristic is similar to buying insurance to cover part of the interest rate risk faced by the FI. The more convex is a given asset, the more insurance against interest rate changes is purchased.

26. A financial institution has an investment horizon of 2 years, 9.5 months. The institution has converted all assets into a portfolio of 8 percent, $1,000, 3-year bonds that are trading at a YTM of 10 percent. The bonds pay interest annually. The portfolio manager believes that the assets are immunized against interest rate changes.

a. Is the portfolio immunized at the time of bond purchase? What is the duration of the bonds?

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Three-year Bonds | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.08 |  | Annual payments | | |
|  | **YTM =** | **0.10** |  |  | Maturity = | 3 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $80.00 | 0.90909 | $72.73 |  | $72.73 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $80.00 | 0.82645 | $66.12 |  | $132.23 |  |  |  |  |  |
| 3 | $1,080 | 0.75131 | $811.42 |  | $2,434.26 |  |  |  |  |  |
|  |  | Price = | $950.26 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $2,639.22 | Duration | = | 2.7774 | = Numerator/Price | |

The bonds have a duration of 2.7774 years, which is 33.33 months. For practical purposes, the bond investment horizon was immunized at the time of purchase.

b. Will the portfolio be immunized one year later?

After one year, the investment horizon will be 1 year, 9.5 months. At this time, the bonds will have a duration of 1.9247 years, or 1 year, 11+ months. Thus the bonds will no longer be immunized.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Two-year Bonds | |  |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.08 |  | Annual payments | | |
|  | **YTM =** | **0.10** |  |  | Maturity = | 3 |  |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T |  |  |  |  |  |
| 1 | $80.00 | 0.90909 | $72.73 |  | $72.73 | PVIF = 1/(1+YTM)^(Time) | | | |  |
| 2 | $1,080 | 0.82645 | $892.56 |  | $1,785.12 |  |  |  |  |  |
|  |  | Price = | $965.29 |  |  |  |  |  |  |  |
|  |  |  | Numerator = | | $1,857.85 | Duration | = | 1.9247 | = Numerator/Price | |

c. Assume that one-year, 8 percent zero-coupon bonds are available in one year. What proportion of the original portfolio should be placed in zeros to rebalance the portfolio?

The investment horizon is 1 year, 9.5 months, or 21.5 months. Thus, the proportion of bonds that should be placed in the zeros can be determined by the following analysis:

21.5 months = X\*12 months + (1-X)\*23 months ⇒ X = 13.6 percent

Thus 13.6 percent of the bond portfolio should be placed in the zeros after one year.

27. MLK Bank has an asset portfolio that consists of $100 million of 30-year, 8 percent coupon, $1,000 bonds that sell at par.

a. What will be the bond’s new prices if market yields change immediately by ± 0.10 percent? What will be the new prices if market yields change immediately by ± 2.00 percent?

At +0.10%: Price = $80\*PVIFAn=30, i=8.1% + $1,000\* PVIFn=30, i=8.1% = $988.85

At –0.10%: Price = $80\*PVIFAn=30, i=7.9% + $1,000\* PVIFn=30, i=7.9% = $1,011.36

At +2.0%: Price = $80\*PVIFAn=30, i=10% + $1,000\* PVIFn=30, i=10% = $811.46

At –2.0%: Price = $80\*PVIFAn=30, i=6.0% + $1,000\* PVIFn=30, i=6.0% = $1,275.30

b. The duration of these bonds is 12.1608 years. What are the predicted bond prices in each of the four cases using the duration rule? What is the amount of error between the duration prediction and the actual market values?

ΔP = -D\*[ΔR/(1+R)]\*P

At +0.10%: ΔP = -12.1608\*0.001/1.08\*$1,000 = -$11.26 ⇒ P’ = $988.74

At -0.10%: ΔP = -12.1608\*-0.001/1.08\*$1,000 = $11.26 ⇒ P’ = $1,011.26

At +2.0%: ΔP = -12.1608\*0.02/1.08\*$1,000 = -$225.20 ⇒ P’ = $774.80

At -2.0%: ΔP = -12.1608\*-0.02/1.08\*$1,000 = $225.20 ⇒ P’ = $1,225.20

Price Price

Market Duration Amount

Determined Estimation of Error

At +0.10%: $988.85 $988.74 $0.11

At -0.10%: $1,011.36 $1,011.26 $0.10

At +2.0%: $811.46 $774.80 $36.66

At -2.0%: $1,275.30 $1,225.20 $50.10

c. Given that convexity is 212.4, what are the bond price predictions in each of the four cases using the duration plus convexity relationship? What is the amount of error in these predictions?

ΔP = {-D\*[ΔR/(1+R)] + ½\*CX\*(ΔR)2}\*P

At +0.10%: ΔP = {-12.1608\*0.001/1.08 + 0.5\*212.4\*(0.001)2}\*$1,000 = -$11.15

At -0.10%: ΔP = {-12.1608\*-0.001/1.08 + 0.5\*212.4\*(-0.001)2}\*$1,000 = $11.366

At +2.0%: ΔP = {-12.1608\*0.02/1.08 + 0.5\*212.4\*(0.02)2}\*$1,000 = -$182.72

At -2.0%: ΔP = {-12.1608\*-0.02/1.08 + 0.5\*212.4\*(-0.02)2}\*$1,000 = $267.68

ΔPrice Price

Price Duration & Duration &

Market Convexity Convexity Amount

Determined Estimation Estimation of Error

At +0.10%: $988.85 -$11.15 $988.85 $0.00

At -0.10%: $1,011.36 $11.37 $1,011.37 $0.01

At +2.0%: $811.46 -$182.72 $817.28 $5.82

At -2.0%: $1,275.30 $267.68 $1,267.68 $7.62

d. Diagram and label clearly the results in parts (a), (b) and (c).

The profiles for the estimates based on only ± 0.10 percent changes in rates are very close together and do not show clearly in a graph. However, the profile relationship would be similar to that shown above for the ± 2.0 percent changes in market rates.

28. Estimate the convexity for each of the following three bonds which all trade at YTM of 8 percent and have face values of $1,000.

A 7-year, zero-coupon bond.

A 7-year, 10 percent annual coupon bond.

A 10-year, 10 percent annual coupon bond that has a duration value of 6.994 (≅ 7) years.

ΔMarket Value ΔMarket Value Capital Loss + Capital Gain

at 8.01 percent at 7.99 percent Divided by Original Price

7-year zero -0.37804819 0.37832833 0.00000048

7-year coupon -0.55606169 0.55643682 0.00000034

10-year coupon -0.73121585 0.73186329 0.00000057

Convexity = 108 \* (Capital Loss + Capital Gain) ÷ Original Price at 8.00 percent

7-year zero CX = 100,000,000\*0.00000048 = 48

7-year coupon CX = 100,000,000\*0.00000034 = 34

10-year coupon CX = 100,000,000\*0.00000057 = 57

An alternative method of calculating convexity for these three bonds using the following equation is illustrated at the end of this problem and onto the following page.



Rank the bonds in terms of convexity, and express the convexity relationship between zeros and coupon bonds in terms of maturity and duration equivalencies.

Ranking, from least to most convexity: 7-year coupon bond, 7-year zero, 10-year coupon

Convexity relationships:

Given the same yield-to-maturity, a zero-coupon bond with the same maturity as a coupon bond will have more convexity.

Given the same yield-to-maturity, a zero-coupon bond with the same duration as a coupon bond will have less convexity.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Zero Coupon Bond | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0 |  |  |  |
|  | YTM = | 0.08 |  |  | Maturity = | 7 |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T | \*(1+T) |  | \*(1+R)^2 |  |
| 1 | $0.00 | 0.92593 | $0.00 |  | $0.00 | $0.00 |  |  |  |
| 2 | $0.00 | 0.85734 | $0.00 |  | $0.00 | $0.00 |  |  |  |
| 3 | $0.00 | 0.79383 | $0.00 |  | $0.00 | $0.00 |  |  |  |
| 4 | $0.00 | 0.73503 | $0.00 |  | $0.00 | $0.00 |  |  |  |
| 5 | $0.00 | 0.68058 | $0.00 |  | $0.00 | $0.00 |  |  |  |
| 6 | $0.00 | 0.63017 | $0.00 |  | $0.00 | $0.00 |  |  |  |
| 7 | 1,000.00 | 0.58349 | $583.49 |  | 4,084.43 | 32,675.46 |  |  |  |
|  |  | Price = | $583.49 |  |  | 32,675.46 |  | 680.58 |  |
|  |  |  | Numerator | = | $4,084.43 | Duration | = | 7.0000 |  |
|  |  |  |  |  |  | Convexity | = | 48.011 |  |
|  | 7-year Coupon Bond | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.1 |  |  |  |
|  | YTM = | 0.08 |  |  | Maturity = | 7 |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T | \*(1+T) |  | \*(1+R)^2 |  |
| 1 | $100.00 | 0.925926 | $92.59 |  | $92.59 | 185.19 |  |  |  |
| 2 | $100.00 | 0.85734 | $85.73 |  | $171.47 | 514.40 |  |  |  |
| 3 | $100.00 | 0.79383 | $79.38 |  | $238.15 | 952.60 |  |  |  |
| 4 | $100.00 | 0.73503 | $73.50 |  | $294.01 | 1,470.06 |  |  |  |
| 5 | $100.00 | 0.68058 | $68.06 |  | $340.29 | 2,041.75 |  |  |  |
| 6 | $100.00 | 0.63017 | $63.02 |  | $378.10 | 2,646.71 |  |  |  |
| 7 | 1,100.00 | 0.58349 | $641.84 |  | $4,492.88 | 35,943.01 |  |  |  |
|  |  | Price = | $1,104.13 |  |  | 43,753.72 |  | 1287.9 |  |
|  |  |  | Numerator | = | $6,007.49 | Duration | = | 5.4409 |  |
|  |  |  |  |  |  | Convexity | = | 33.974 |  |
|  | 10-year Coupon Bond | |  |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.1 |  |  |  |
|  | YTM = | 0.08 |  |  | Maturity = | 10 |  |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T | \*(1+T) |  | \*(1+R)^2 |  |
| 1 | $100.00 | 0.925926 | $92.59 |  | $92.59 | 185.19 |  |  |  |
| 2 | $100.00 | 0.857339 | $85.73 |  | $171.47 | 514.40 |  |  |  |
| 3 | $100.00 | 0.793832 | $79.38 |  | $238.15 | 952.60 |  |  |  |
| 4 | $100.00 | 0.735030 | $73.50 |  | $294.01 | 1470.06 |  |  |  |
| 5 | $100.00 | 0.680583 | $68.06 |  | $340.29 | 2041.75 |  |  |  |
| 6 | $100.00 | 0.630170 | $63.02 |  | $378.10 | 2646.71 |  |  |  |
| 7 | $100.00 | 0.583490 | $58.35 |  | $408.44 | 3267.55 |  |  |  |
| 8 | $100.00 | 0.540269 | $54.03 |  | $432.22 | 3889.94 |  |  |  |
| 9 | $100.00 | 0.500249 | $50.02 |  | $450.22 | 4502.24 |  |  |  |
| 10 | $1,100.0 | 0.463193 | $509.51 |  | 5,095.13 | 56046.41 |  |  |  |
|  |  | Price = | 1,134.20 |  |  | 75516.84 |  | 1322.9 |  |
|  |  |  | Numerator | = | 7,900.63 | Duration | = | 6.9658 |  |
|  |  |  |  |  |  | Convexity | = | 57.083 |  |

29. A 10-year, 10 percent annual coupon, $1,000 bond trades at a YTM of 8 percent. The bond has a duration of 6.994 years. What is the modified duration of this bond? What is the practical value of calculating modified duration? Does modified duration change the result in using the duration relationship to estimate price sensitivity?

Modified duration = Duration/(1+ R) = 6.994/1.08 = 6.4759. Some practitioners find this value easier to use because the percentage change in value can be estimated simply by multiplying the existing value times the basis point change in interest rates rather than by the relative change in interest rates. Using modified duration will not change the estimated price sensitivity of the asset.

Additional Examples for Chapter 7This example is to estimate both the duration and convexity of a 6-year bond paying 5 percent coupon annually and the annual yield to maturity is 6 percent.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 6-year Coupon Bond | |  |  |  |  |  |  |
|  | Par value = | $1,000 |  |  | Coupon = | 0.05 |  |  |
|  | YTM = | 0.06 |  |  | Maturity = | 6 |  |  |
| Time | Cash Flow | PVIF | PV of CF |  | PV\*CF\*T | \*(1+T) |  | \*(1+R)^2 |
| 1 | $50.00 | 0.94340 | $47.17 |  | $47.17 | $94.34 |  |  |
| 2 | $50.00 | 0.89000 | $44.50 |  | $89.00 | $267.00 |  |  |
| 3 | $50.00 | 0.83962 | $41.98 |  | $125.94 | $503.77 |  |  |
| 4 | $50.00 | 0.79209 | $39.60 |  | $158.42 | $792.09 |  |  |
| 5 | $50.00 | 0.74726 | $37.36 |  | $186.81 | 1,120.89 |  |  |
| 6 | $1,050.00 | 0.70496 | $740.21 |  | $4,441.25 | 31,088.76 |  |  |
|  |  | Price = | $950.83 |  |  | 33,866.85 |  | 1068.3 |
|  |  |  | Numerator | = | $5,048.60 | Duration | = | 5.3097 |
|  |  |  |  |  |  | Convexity | = | 31.7 |

Using the textbook method:

CX = 108 [(950.3506-950.8268)/950.8268 + (951.3032-950.8268)/950.8268]

= 108[-0.0005007559 + 0.0005501073] = 31.70

What is the effect of a 2 percent increase in interest rates, from 6 percent to 8 percent?

Using Present Values, the percentage change is:

= ($950.8268 - $861.3136)/ $950.8268 = -9.41%

Using the duration formula: MVA = -D\*R/(1 + R) + 0.5CX(R)2

= -5.3097\*[(0.02)/1.06] + 0.5(31.7)(0.02)2

= -0.1002 + .0063 = -9.38%

Adding convexity adds more precision. Duration alone would have given the answer of -10.02%.